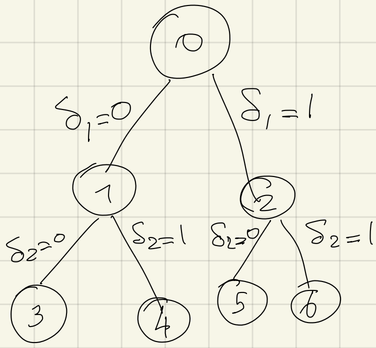
Exam optimization 23-02-2023

Exercise 1

1. Rewrite the optimization problem in **standard form**. Depict the tree associated to the MILP.

$$\min \ \delta_2 + 3x_1 \qquad\qquad \min \ \delta_2 + 3x_1$$
$$0.5\delta_1 + \delta_2 \geq 1 \qquad\qquad 0.5\delta_1 + \delta_2 - s_1 = 1$$
$$x_1 \geq \delta_2 \qquad\qquad -x_1 + \delta_2 + s_2 = 0$$
$$\delta_1, \delta_2 \in \{0,1\} \qquad\qquad \delta_1, \delta_2 \in \{0,1\}$$
$$x_1 \geq 0 \qquad\qquad x_1, s_1, s_2 \geq 0$$



$\delta_1 = 0 \rightarrow$

$$\min \ \delta_2 + 3x_1 \qquad\qquad \min \ \delta_2 + 3x_1$$
$$\delta_2 \geq 1 \quad \longrightarrow \quad \delta_2 - s_1 = 1$$
$$x_1 \geq \delta_2 \qquad\qquad \delta_2 - x_1 + s_2 = 0$$
$$\delta_1, \delta_2 \in \{0,1\} \qquad\qquad \delta_2 + s_3 = 1$$
$$x_1 \geq 0 \qquad\qquad x_1, \delta_2, s_1, s_2, s_3 \geq 0$$

$$\min \ y_1 + y_2 + y_3$$
$$\delta_2 - s_1 + y_1 = 1$$
$$\delta_2 - x_1 + s_2 + y_2 = 0$$
$$\delta_2 + s_3 + y_3 = 1$$
$$x_1, \delta_2, s_1, s_2, s_3, y_1, y_2, y_3 \geq 0$$

$\longrightarrow$

		x_1	δ_2	s_1	s_2	s_3	y_1	y_2	y_3
	0	0	0	0	0	0	1	1	1
	1	0	1	-1	0	0	1	0	0
s_2	0	-1	1	0	1	0	0	1	0
s_3	1	0	1	0	0	1	0	0	1

Subtract all rows to 1st one

			x_1	δ_2	s_1	s_2	s_3	y_1	y_2	y_3	
		-2	1	-3	1	-1	-1	0	0	0	+ 3AUX
1/1	y_1	1	0	1	-1	0	0	1	0	0	- AUX
0/1	y_2	0	-1	1	0	1	0	0	1	0	
1/1	y_3	1	0	1	0	0	1	0	0	1	- AUX

		x_1	δ_2	s_1	s_2	s_3	y_1	y_2	y_3	
	-2	-2	0	1	2	-1	0	3	0	$+2AUX$
y_1	1	1	0	-1	-1	0	1	-1	0	$=AUX$
δ_2	0	-1	1	0	1	0	0	1	0	$+AUX$
y_3	1	1	0	0	-1	1	0	-1	1	$-AUX$

		x_1	δ_2	s_1	s_2	s_3	y_1	y_2	y_3	
	0	0	0	-1	0	-1	2	1	0	$+AUX$
x_1	1	1	0	-1	-1	0	1	-1	0	$+AUX$
δ_2	1	0	1	-1	0	0	1	0	0	$+AUX$
y_3	0	0	0	1	0	1	-1	0	1	$=AUX$

		x_1	δ_2	s_1	s_2	s_3	y_1	y_2	y_3
	0	0	0	0	0	0	1	1	1
x_1	1	1	0	0	-1	1	0	-1	1
δ_2	1	0	1	0	0	1	0	0	1
s_1	0	0	0	1	0	1	-1	0	1

Ph 1 OK

Ph 2 Min

		x_1	δ_2	s_1	s_2	s_3	
	0	3	1	0	0	0	$-3AUX$
	1	1	0	0	-1	1	$=AUX$
	1	0	1	0	0	1	
s_1	0	0	0	1	0	1	

		x_1	δ_2	s_1	s_2	s_3	
	-3	0	1	0	3	-3	$-AUX$
x_1	1	1	0	0	-1	1	
	1	0	1	0	0	1	$=AUX$
s_1	0	0	0	1	0	1	

		x_1	δ_2	s_1	s_2	s_3	
	-4	0	0	0	3	-4	$+4AUX$
x_1	1	1	0	0	-1	1	$-AUX$
δ_2	1	0	1	0	0	1	$-AUX$
s_1	0	0	0	1	0	1	$=AUX$

	-4	x_1: 0	δ_2: 0	s_1: +4	s_2: 3	s_3: 0
x_1	1	1	0	-1	-1	0
δ_2	1	0	1	-1	0	0
s_3	0	0	0	1	0	1

Ph 2 over

$J^* = 4$

$$x = \begin{bmatrix} x_1 \\ \delta_1 \\ \delta_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

feasible → incumbent solution

$\delta_1 = 1$

$$\begin{aligned} \min\ & \delta_2 + 3x_1 \\ & \delta_2 \geq 0.5 \\ & x_1 \geq \delta_2 \\ & \delta_2 \in \{0,1\} \\ & x_1 \geq 0 \end{aligned} \quad \xrightarrow{\text{Relax}} \text{+ standard} \quad \begin{aligned} \min\ & \delta_2 + 3x_1 \\ & \delta_2 - s_1 = 0.5 \\ & \delta_2 - x_1 + s_2 = 0 \\ & \delta_2 + s_3 = 1 \\ & x_1, \delta_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Ph. I

$$\begin{aligned} \min\ & y_1 + y_2 + y_3 \\ & \delta_2 - s_1 + y_1 = 0.5 \\ & \delta_2 - x_1 + s_2 + y_2 = 0 \\ & \delta_2 + s_3 + y_3 = 1 \\ & x_1, \delta_2, s_1, s_2, s_3, y_1, y_2, y_3 \geq 0 \end{aligned}$$

	0	x_1: 0	δ_2: 0	s_1: 0	s_2: 0	s_3: 0	y_1: $\cancel{1}$	y_2: $\cancel{1}$	y_3: $\cancel{1}$
	0.5	0	1	-1	0	0	1	0	0
s_2	0	-1	1	0	1	0	0	1	0
s_3	1	0	1	0	0	1	0	0	1

subtract all rows to 1st one

		x_1	δ_2	s_1	s_2	s_3	y_1	y_2	y_3	
	-1.5	1	-3	1	-1	-1	0	0	0	+3AUX
y_1	0.5	0	1	-1	0	0	1	0	0	-AUX
y_2	0	-1	1	0	1	0	0	1	0	=AUX
y_3	1	0	1	0	0	1	0	0	1	-AUX

		x_1	δ_2	s_1	s_2	s_3	y_1	y_2	y_3	
	-1.5	-2	0	1	2	-1	0	3	0	+2AUX
y_1	0.5	1	0	-1	-1	0	1	-1	0	=AUX
δ_2	0	-1	1	0	1	0	0	1	0	+AUX
y_3	1	1	0	0	-1	1	0	-1	1	-AUX

		x_1	δ_2	s_1	s_2	s_3	y_1	y_2	y_3	
	-0.5	0	0	-1	0	-1	2	1	0	+AUX
x_1	0.5	1	0	-1	-1	0	1	-1	0	+AUX
δ_2	0.5	0	1	-1	0	0	1	0	0	+AUX
y_3	0.5	0	0	1	0	1	-1	0	1	=AUX

		x_1	δ_2	s_1	s_2	s_3	y_1	y_2	y_3
	0	0	0	0	0	0	1	1	1
x_1	1	1	0	0	-1	1	0	-1	1
δ_2	1	0	1	0	0	1	0	0	1
s_1	0.5	0	0	1	0	1	-1	0	1

Ph.1 OK

Ph.2 min

		x_1	δ_2	s_1	s_2	s_3	
	0	3	1	0	0	0	-3AUX
	1	1	0	0	-1	1	=AUX
	1	0	1	0	0	1	
s_1	0.5	0	0	1	0	1	

		x_1	δ_2	s_1	s_2	s_3	
	-3	0	1	0	$+3$	-3	$-AUX$
x_1	1	1	0	0	-1	1	
	1	0	1	0	0	1	$=AUX$
s_1	0.5	0	0	1	0	1	

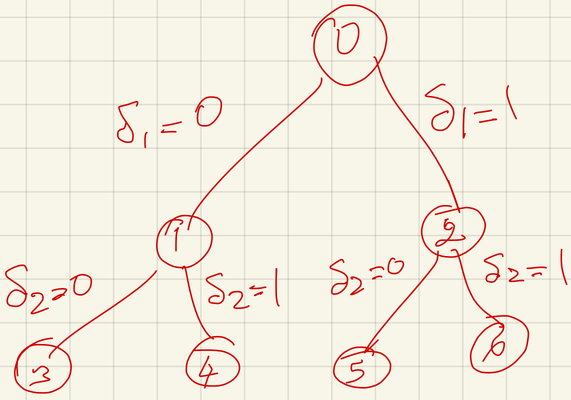
		x_1	δ_2	s_1	s_2	s_3	
	-4	0	0	0	3	-4	$+4AUX$
x_1	1	1	0	0	-1	1	$-AUX$
δ_2	1	0	1	0	0	1	$-AUX$
s_1	0.5	0	0	1	0	1	$=AUX$

		x_1	δ_2	s_1	s_2	s_3
	-2	0	0	4	3	0
x_1	0.5	1	0	-1	-1	0
δ_2	0.5	0	1	-1	0	0
s_3	0.5	0	0	1	0	1

$J^* = 2$

$$x^* = \begin{bmatrix} x_1 \\ \delta_1 \\ \delta_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}$$

Not feasible for the original problem since $\delta_2 = 0.5$



However since $J^* = 2$ is better than the cost of the incumbent solution, it could be possible to encounter a better solution on the branch below node 2. We need to further explore the tree.

3 evenings 5h/evening → x_{TU} # hours Tue; x_{WE} " Wed; x_{TH} " Thu

21 at least 1h/evening

- opt var - cost
- constraints

$x_{TU} \le 5$ $x_{TU} \ge 1$

$x_{WE} \le 5$ $x_{WE} \ge 1$

$x_{TH} \le 5$ $x_{TH} \ge 1$

Cachet 25 K€/h

$$\boxed{25 \cdot 10^3 (x_{TU} + x_{WE} + x_{TH})}$$

Training < SR, MI

δ_{TR}: 0 SR, 1 MI

If $\delta_{TR} = 0$ → 2 nights hotel

4000 · 2

In the cost $\boxed{-8000(1-\delta_{TR})}$

Trips

$\delta_{TR} = 0$			$\delta_{TR} = 1$		
MI - SR	Tue	Paid	MI - SR	Tue	Paid
SR - MI	Thu	Paid	SR - MI	Tue	
MI - SR	Thu		MI - SR	Wed	
SR - MI	Thu	Paid	SR - MI	Wed	
			MI - SR	Thu	
			SR - MI	Thu	Paid

$$\boxed{\delta_{H/C} = \begin{cases} 0 & \text{helicopter} \\ 1 & \text{car} \end{cases}}$$ $\rightarrow 2.2\,k€$ / $\rightarrow 1\,k€$

Therefore we have in the cost function

$$\boxed{-(1-\delta_{TR}) \cdot \left((1-\delta_{H/C}) \cdot 2200 + \delta_{H/C} \cdot 1000 \right) - \delta_{TR} \cdot \underset{\uparrow}{4} \cdot \left((1-\delta_{H/C}) \cdot 2200 + \delta_{H/C} \cdot 1000 \right)}$$

unpaid trips

· 12h/day for Tue - Wed for travel - train & festival

Train at most 8h

$$\boxed{\begin{aligned} x_{TRAIN-TUE} &= \text{\# of train hours TUE} \\ x_{TRAIN-WED} &= \text{" " " WED} \end{aligned}}$$

$$\boxed{\begin{aligned} x_{TRAIN-TUE} &\le 8 \\ x_{TRAIN-WED} &\le 8 \end{aligned}}$$

Travel time

$$\boxed{x_{TU} + x_{TRAIN-TU} + \underbrace{\left(1 \cdot (1-\delta_{H/C}) + 3\delta_{H/C}\right) \cdot \left((1-\delta_{TR}) + 2\delta_{TR}\right)} \le 12}$$

$$\boxed{\cdot\; x_{WE} + x_{TRAIN-WE} + \left(1 \cdot (1-\delta_{H/C}) + 3\delta_{H/C}\right) \cdot 2\delta_{TR} \le 12}$$

· Play only if TRAIN $\ge$ 14h

$$\boxed{\delta_P = \begin{cases} 0 & \text{no play} \\ 1 & \text{play} \end{cases}}$$

$\delta_p = 1 \longleftrightarrow x_{TRAIN-TU} + x_{TRAIN-WED} \geq 14$

Bonus in cost function $\boxed{60k \cdot \delta_p}$

To be translated

$L = -14 \quad U = 8 + 8 - 14 = 2$

$\delta_p = 1 \rightarrow \boxed{x_{TRAIN-TU} + x_{TRAIN-WED} - 14} \geq 0$

$\delta_p = 0 \rightarrow x_{TRAIN-TU} + x_{TRAIN-WED} - 14 \leq -\varepsilon$

$$x_{TRAIN-TU} + x_{TRAIN-WED} - 14 \geq L(1-\delta_p) = -14(1-\delta_p)$$

$$x_{TRAIN-TU} + x_{TRAIN-WED} - 14 \leq (U+\varepsilon)\delta_p - \varepsilon = (2+\varepsilon)\delta_p - \varepsilon$$

OPTIMIZATION PROBLEM

$$\max_{\substack{x_{TU}, x_{WE}, x_{TH} \\ \delta_{TR}, \delta_{H/C} \\ x_{TRAIN-TUE} \\ x_{TRAIN-WED} \\ \delta_p}} 25 \cdot 10^3 (x_{TU} + x_{WE} + x_{TH}) - 8000(1-\delta_{TR}) + $$

$$- (1-\delta_{TR}) \cdot \left((1-\delta_{H/C}) \cdot 2200 + \delta_{H/C} \cdot 1000 \right)$$

$$- \delta_{TR} \cdot 4 \cdot \left((1-\delta_{H/C}) \cdot 2200 + \delta_{H/C} \cdot 1000 \right)$$

$$+ 60 \cdot 10^3 \delta_p$$

subject to

$x_{TU} \leq 5, \; x_{TU} \geq 1$

$x_{WE} \leq 5, \; x_{WE} \geq 1$

$x_{TH} \leq 5, \; x_{TH} \geq 1$

$x_{TRAIN-TUE} \leq 8$

$x_{TRAIN-WED} \leq 8$

$$x_{TU} + x_{TRAIN-TU} + \left(1\cdot(1-\delta_{H/C}) + 3\delta_{H/C}\right)\cdot\left((1-\delta_{TR}) + 2\delta_{TR}\right) \le 12$$

$$x_{WE} + x_{TRAIN-WE} + \left(1\cdot(1-\delta_{H/C}) + 3\delta_{H/C}\right)\cdot 2\delta_{TR} \le 12$$

$$x_{TRAIN-TU} + x_{TRAIN-WED} - 14 \ge -14(1-\delta_P)$$

$$x_{TRAIN-TU} + x_{TRAIN-WED} - 14 \le -(2+\varepsilon)\delta_P - \varepsilon$$

$$x_{TU}, x_{WE}, x_{TH}, x_{TRAIN-TUE}, x_{TRAIN-WED} \ge 0$$

$$\delta_{TR}, \delta_{H/C}, \delta_P \in \{0,1\}$$

Ex. 3

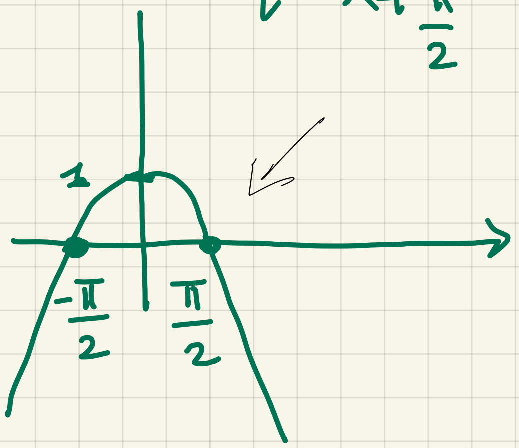
$$\max f(x)$$

$$x^2 \ge 0$$

$$x \le -1$$

$$x \ge +1$$

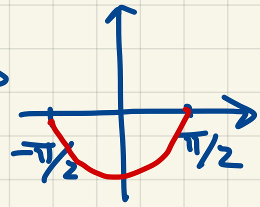
$$f(x) = \begin{cases} x + \frac{\pi}{2} & x < -\frac{\pi}{2} \\ \cos(x) & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ -x + \frac{\pi}{2} & x > \frac{\pi}{2} \end{cases}$$



Analysis of convexity of $f(x)$

$$\nabla^2 f(x) = \begin{cases} 0, & x < -\frac{\pi}{2} \quad \ge 0 \text{ ok convex/concave} \\ -\cos(x) & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \quad \le 0 \text{ not convex} \\ 0, & x > \frac{\pi}{2} \quad \ge 0 \text{ convex/concave} \end{cases}$$

↓ concave!



So, it is definitely not convex.

Let's check if concave

$$e^- = \frac{d}{dx} f(x) \Big|_{x=-\frac{\pi}{2}^-} = 1$$

$$e^+ = \frac{d}{dx} f(x) \Big|_{x=-\frac{\pi}{2}^+} = -\sin\left(-\frac{\pi}{2}\right) = 1$$

$e^- = e^+ \rightarrow$ valid for both convexity and concavity

$$e^- = \frac{d}{dx} f(x) \Big|_{x=\frac{\pi}{2}^-} = -\sin\frac{\pi}{2} = -1$$

$$e^+ = \frac{d}{dx} f(x) \Big|_{x=\frac{\pi}{2}^+} = -1$$

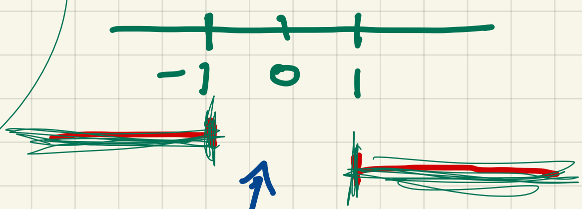
Again $\ell^- = \ell^+$

Notice that the 3 functions are concave and $\ell^- = \ell^+$ for $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$. For concavity we need that $\ell^- \geq \ell^+$ plus the concavity of the 3 functions

$\longrightarrow$ the function is concave

2) $\boxed{x^2 \geq 0} \longrightarrow \underline{\forall x \in \mathbb{R}}$

- $x \leq -1$
- $x \geq +1$

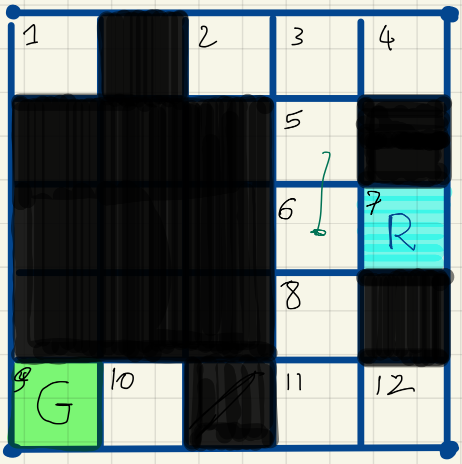


the intersection is empty!

The empty set is convex

3) Since we maximize a concave function and the feas set is convex $\rightarrow$ the problem is convex

Ex. 4



4.1 There is no way to attain G for any initial condition $N=\infty$

$$J_1(x) = \begin{cases} \infty & x_1 = 1 \\ \underset{R}{1} + \infty = \infty & x_1 = 2 \\ \min\left(\underset{D}{1+\infty}, \underset{L}{1+\infty}, \underset{R}{1+\infty}\right) = \infty & x_1 = 3 \\ 1+\infty = \infty & x_1 = 4 \\ \min\left(\underset{U}{1}+\infty, \underset{D}{1}+\infty\right) = \infty & x_1 = 5 \\ \min\left(\underset{U}{1}+\infty, \underset{D}{1+\infty}, \underset{R}{1+\infty}\right) = \infty & x_1 = 6 \\ 1+\infty = \infty & x_1 = 7 \\ \min\left(\underset{U}{1+\infty}, \underset{D}{1+\infty}\right) = \infty & x_1 = 8 \\ \min\left(\underset{R}{1+\infty}, \underset{S}{0.5+0}\right) = 0.5 & x_1 = 9 \\ 1+0 = 1 & x_1 = 10 \\ \min\left(\underset{U}{1+\infty}, \underset{R}{1+\infty}\right) = \infty & x_1 = 11 \\ 1+\infty = \infty & x_1 = 12 \end{cases}$$

$$\mu_1(x) = \begin{cases} / & x_1 = 1 \\ R & x_1 = 2 \\ D/L/R & x_1 = 3 \\ L & x_1 = 4 \\ U/D & x_1 = 5 \\ U/D/R & x_1 = 6 \\ L & x_1 = 7 \\ U/D & x_1 = 8 \\ S & x_1 = 9 \\ L & x_1 = 10 \\ U/R & x_1 = 11 \\ L & x_1 = 12 \end{cases}$$

	U	D	L	R	S
1	/	/	/	/	/
2	/	/	/	1	/
3	/	1	1	1	/
4	/	/	1	/	/
5	1	1	/	/	/
6	1	1	/	1	/
7	/	/	1	/	/
8	1	1	/	/	/
9	/	/	/	1	1
10	/	/	1	/	/
11	1	/	/	1	/
12	/	/	1	/	/

$$g_2(x) = \begin{cases} \infty & x_2 = 1 \\ \infty & x_2 = 2 \\ \infty & x_2 = 3 \\ \infty & x_2 = 4 \\ \infty & x_2 = 5 \\ \infty & x_2 = 6 \\ \infty & x_2 = 7 \\ \infty & x_2 = 8 \\ 0 & x_2 = 9 \\ \infty \quad 1 & x_2 = 10 \\ \infty & x_2 = 11 \\ \infty & x_2 = 12 \end{cases}$$